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On the operator bases underlying Wigner's, Kirkwood's and Glauber's phase space functions

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Abstract. The Hermitian operators behind Wigner's phase space function are recognised to be simple ordered exponentials of the dynamical variables. This operator basis is highly symmetric; it supplies a perfectly unbiased formulation of operator equations in terms of phase space functions, which is as close to classical physics as it possibly can be. We demonstrate how the symmetry properties of the basis can be exploited to simplify the computation of Wigner functions enormously. The ordered-operator methods are also applicable to the phase space functions of Kirkwood and Glauber type. Both kinds are briefly discussed with emphasis on how they differ from Wigner's description.

1. Introduction

Wigner's phase space function, introduced in 1932 as a tool for studying quantum corrections to classical equilibrium distributions [1] (but actually found some years earlier 'for another purpose'†), has been studied extensively and applied to a large variety of problems (for a recent review see [4]; various applications are discussed in [5]). In particular, the Wigner function is known to possess remarkable symmetry properties, some of which can be used for a unique characterisation [6].

In this paper we show that all these various symmetry properties are implications of the invariance, under arbitrary linear canonical transformations, of the Hermitian operator basis underlying Wigner's phase space function. This invariance, stated in (10) below, signifies that the corresponding injunction for translating statements about operators into equations of numerical functions is perfectly unbiased: no point, no direction, no scale is distinguished from any other in phase space, and Hermitian dynamical variables are not preferred to their non-Hermitian alternatives.

In contrast, other popular phase space functions *are* biased. For instance, Kirkwood's function [7] specifies a direction in phase space, and Glauber's P function [8] fixes a scale. These two are representative examples of two classes of phase space functions, each class containing Wigner's function.

The second section of this paper deals with Wigner's phase space function. We demonstrate that the operators of the Hermitian basis are simple ordered exponentials. This observation, combined with the above-mentioned invariance of the operator basis,

† See the somewhat mysterious footnote 2 in [1]. It should also be mentioned that in 1930 Dirac [2] (in his seminal paper about introducing exchange corrections into Thomas-Fermi theory) made use of 'corresponding classical functions' which are essentially identical to Wigner's function. Further, in 1931 Heisenberg was led to consider [3] a Wigner function when investigating incoherent scattering of x-rays off atoms.

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can be employed to simplify the computation of Wigner functions enormously. We illustrate the power of our new formalism by a few examples. Naturally, the emphasis is on the novel approach; encyclopedic listings of properties of the Wigner functions are already available in the literature (see e.g. [4, 9]). We shall, however, use the new insights for concise derivations of the fundamental equations obeyed by Wigner functions. A couple of properties, not as yet on record, are found as an additional bonus. Further, we show that, as a consequence of the total lack of bias stated above, Wigner's phase space description is as close to classical dynamics as it possibly can be: whenever Heisenberg's equations of motion are formally identical to the classical Hamilton equations, the Wigner function of the density operator obeys the classical Liouville theorem. It is for this reason that Wigner's phase space function is the most natural starting point for studying quantum corrections to (semi-)classical treatments.

Section 3 is concerned with Kirkwood-type scale-invariant descriptions, which are biased towards certain directions in phase space and towards Hermitian dynamical variables. Less detail is given here, since we are mainly interested in pointing out the more important respects in which Kirkwood's function is similar to Wigner's, and where the two functions differ significantly.

In § 4 we address Glauber-type rotationally invariant descriptions. These select a certain scale in phase space and appear more naturally in terms of non-Hermitian dynamical variables. Inasmuch as these are not quite on equal footing with the Hermitian variables, certain problems arise which require more careful considerations. In particular, some of the sets of operators, including those behind Glauber's popular P function, are incomplete; they do not really provide a basis in operator space. Examples of simple operators which cannot be expanded in these 'bases' are presented.

2. Wigner's operator basis

It suffices to consider one continuous degree of freedom, the generalisation to many being immediate. After picking an arbitrary scale for distances and the corresponding scale for momenta, the Hermitian dynamical variables are the dimensionless position and momentum operators q and p with the commutator $[q, p] = \mathrm{i}$; all operators are functions of q and p. Wigner's phase space function of an operator $F(q, p)$ is then defined by

$$F_{\mathrm{W}}(q', p') = \int \mathrm{d}s \langle q' - \tfrac{1}{2}s | F(q, p) | q' + \tfrac{1}{2}s \rangle \exp(\mathrm{i}p's) \tag{1}$$

where we do not include a factor of $1/2\pi$, as most authors do. Instead of matrix elements referring to position only, it is more useful to employ mixed q, p or p, q matrix elements. Then we have the equivalent expressions

$$\begin{aligned} F_{\mathrm{W}}(q', p') &= \int \mathrm{d}q''\, \mathrm{d}p'' \langle q'' | F(q, p) | p'' \rangle \langle p'' | q'' \rangle 2 \exp[2\mathrm{i}(p'' - p')(q'' - q')] \\ &= \int \mathrm{d}q''\, \mathrm{d}p'' \langle p'' | F(q, p) | q'' \rangle \langle q'' | p'' \rangle 2 \exp[-2\mathrm{i}(q'' - q')(p'' - p')]. \end{aligned} \tag{2}$$

The observation that, for example,

$$\langle p'' | q'' \rangle 2 \exp[2\mathrm{i}(p'' - p')(q'' - q')] = \langle p'' | 2 \exp[2\mathrm{i}(p - p'); (q - q')] | q'' \rangle \tag{3}$$

where we make use of Schwinger's notation [10] for ordered exponentials

$$\exp(A;\, B) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k B^k \tag{4}$$

now shows that

$$\begin{aligned} F_{\mathrm{W}}(q', p') &= \mathrm{Tr}\{F(q, p) 2 \exp[2\mathrm{i}(p-p');\, (q-q')]\} \\ &= \mathrm{Tr}[F(q+q', p+p') 2 \exp(2\mathrm{i}p;\, q)] \end{aligned} \tag{5a}$$

or, from the second version in (2),

$$\begin{aligned} F_{\mathrm{W}}(q', p') &= \mathrm{Tr}\{F(q, p) 2 \exp[-2\mathrm{i}(q-q');\, (p-p')]\} \\ &= \mathrm{Tr}[F(q+q', p+p') 2 \exp(-2\mathrm{i}q;\, p)]. \end{aligned} \tag{5b}$$

These equations can be inverted, so that the operator is given in terms of its Wigner function by

$$\begin{aligned} F(q, p) &= \int \frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi} F_{\mathrm{W}}(q', p') 2 \exp[2\mathrm{i}(p-p');\, (q-q')] \\ &= \int \frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi} F_{\mathrm{W}}(q', p') 2 \exp[-2\mathrm{i}(q-q');\, (p-p')]. \end{aligned} \tag{6}$$

For an elementary proof of this statement, evaluate $\langle p''|F(q, p)|q''\rangle$.

Equations (5) and (6) together signify the completeness of the operator basis

$$2 \exp[2\mathrm{i}(p-p');\, (q-q')] = 2 \exp[-2\mathrm{i}(q-q');\, (p-p')]. \tag{7}$$

This equality states that these operators are Hermitian. We have thus identified the Hermitian operator basis underlying Wigner's phase space function. The Wigner function itself is, therefore, the coefficient function that appears when the operator is expanded in the basis (7). As such it is the analogue of a wavefunction that describes a Hilbert-space state with respect to a Hilbert-space basis.

The individual operators (7) are obtained from their 'seed' $2 \exp(2\mathrm{i}p;\, q)$ by translations in phase space,

$$2 \exp[2\mathrm{i}(p-p');\, (q-q')] = \exp[-\mathrm{i}(pq'-p'q)] 2 \exp(2\mathrm{i}p;\, q) \exp[\mathrm{i}(pq'-p'q)]. \tag{8}$$

As a consequence, Wigner's function is translationally invariant in the sense that if $\tilde{F}(q, p) = F(q-q'', p-p'')$ with numbers q'' and p'', then $\tilde{F}_{\mathrm{W}}(q', p') = F_{\mathrm{W}}(q'-q'', p'-p'')$. Further, because of (8) it suffices to study the properties of the seed, the properties of an arbitrary basis operator follow immediately.

Here then is the fundamental symmetry property of the seed: if, for complex numbers α, β, σ, and τ,

$$Q = \alpha q + \beta p \qquad P = \sigma q + \tau p \tag{9}$$

are such that $[Q, P] = \mathrm{i}$, i.e. $\alpha\tau - \beta\sigma = 1$, then

$$2 \exp(2\mathrm{i}P;\, Q) = 2 \exp(2\mathrm{i}p;\, q) \tag{10}$$

provided that P has left eigenstates $\langle P'|P = P'\langle P'|$ and Q has right eigenstates $Q|Q'\rangle = |Q'\rangle Q'$, which requires $\mathrm{Im}(\alpha^*\beta) \geqslant 0$ and $\mathrm{Im}(\sigma\tau^*) \geqslant 0$. The proof of this statement employs the transformation functions

$$\begin{aligned} \langle q'|Q''\rangle &= C_{qQ} \exp[-(\mathrm{i}/\beta)(\tfrac{1}{2}\alpha q'^2 - q'Q'' + \tfrac{1}{2}\tau Q''^2)] \\ \langle P''|p'\rangle &= C_{Pp} \exp[-(\mathrm{i}/\sigma)(\tfrac{1}{2}\alpha P''^2 - P''p' + \tfrac{1}{2}\tau p'^2)] \end{aligned} \tag{11}$$

where the constants C_{qQ} and C_{Pp} reflect the specific choice of phase and normalisation, in evaluating

$$\langle P''|Q''\rangle = (1/\mathrm{i})(2\pi\beta\sigma)^{1/2} C_{Pp} C_{qQ} \exp(-\mathrm{i}P''Q'') \tag{12}$$

and

$$\begin{aligned}\langle P''|2\exp(2\mathrm{i}p;q)|Q''\rangle &= (2/\mathrm{i})(2\pi\beta\sigma)^{1/2} C_{Pp} C_{qQ} \exp(\mathrm{i}P''Q'') \\ &= \langle P''|Q''\rangle 2\exp(2\mathrm{i}P''Q'') \\ &= \langle P''|2\exp(2\mathrm{i}P;\,Q)|Q''\rangle \end{aligned}\tag{13}$$

which establishes (10).

For $\alpha = \tau = 0$, $\beta = -\sigma = 1$, equation (10) in conjunction with (8) reproduces (7). Other important examples are (i) scaling transformations $Q = \mathrm{e}^{z}q$, $P = \mathrm{e}^{-z}p$; (ii) rotations in phase space $Q = q\cos\phi + p\sin\phi$, $P = p\cos\phi - q\sin\phi$; (iii) introduction of non-Hermitian dynamical variables ('creation and annihilation operators' or 'ladder operators')

$$Q = (1/\sqrt{2})(q + \mathrm{i}p) \equiv y \qquad P = (1/\sqrt{2})(p + \mathrm{i}q) = \mathrm{i}y^{\dagger} \tag{14}$$

so that

$$2\exp(2\mathrm{i}p;\,q) = 2\exp(-2y^{\dagger};\,y) \tag{15}$$

but *not* $Q = \mathrm{i}y^{\dagger}$, $P = -y$ since the resulting seed $2\exp(2y;\,y^{\dagger})$ has an empty domain (see below). Here the requirement that Q have right eigenstates and P left ones is essential.

In terms of the Wigner functions, the invariance (10) says that if $\tilde{F}(q,p) = F(\alpha q + \beta p, \sigma q + \tau p)$ then $\tilde{F}_{\mathrm{W}}(q',p') = F_{\mathrm{W}}(\alpha q' + \beta p', \sigma q' + \tau p')$. To illustrate how this insight can be used to simplify calculations, we evaluate the Wigner function of the projection operator to a generalised minimum uncertainty state (a 'squeezed' state), a problem that is the subject of a recent publication [11]. This operator is

$$\tilde{F}(q,p) = \exp(-Y^{\dagger};\,Y) = F(Y, \mathrm{i}Y^{\dagger}) \tag{16}$$

where (modulo irrelevant translations)

$$Y = (1/\sqrt{2})[\mathrm{e}^{z}(q\cos\phi + p\sin\phi) + \mathrm{i}\,\mathrm{e}^{-z}(p\cos\phi - q\sin\phi)] \tag{17}$$

is obtained when the transition to non-Hermitian operators (iii) is performed after rotating (ii) and changing the scale (i). The Wigner function of

$$F(q,p) = \exp(\mathrm{i}p;\,q) = 2\pi\delta(q)\delta(p) \tag{18}$$

is†

$$\begin{aligned} F_{\mathrm{W}}(q',p') &= \mathrm{Tr}\{2\pi\delta(q)\delta(p)2\exp[2\mathrm{i}(p-p');\,(q-q')]\} \\ &= \int \frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi} 2\pi\delta(q'')\delta(p'')2\exp[2\mathrm{i}(p''-p')(q''-q')] \\ &= 2\exp(2\mathrm{i}p'q') \end{aligned}\tag{19}$$

† This is an illustration of the fact that the trace of an *ordered* operator $F(q,p)$ (all the q to the left of all the p or vice versa) is given by the classical phase space integral

$$\mathrm{Tr}\,F(q,p) = \int \frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi} F(q',p')$$

which, in conjunction with the cyclic property of traces, is a frequently used tool.

so that for $q \to Q = Y$ and $p \to P = \mathrm{i}Y^\dagger$ we get

$$\tilde{F}_W(q', p') = 2\exp[-e^{2z}(q'\cos\phi + p'\sin\phi)^2 - e^{-2z}(p'\cos\phi - q'\sin\phi)^2]. \tag{20}$$

We continue the discussion of the properties of the seed (10) by remarking that the rotational invariance (ii) implies that the seed commutes with $y^\dagger y$, the generator of these rotations. Consequently, the stationary uncertainty states $|n\rangle$, which obey

$$\begin{aligned} &y^\dagger y|n\rangle = |n\rangle n \qquad y^k|n\rangle = |n-k\rangle(n!/(n-k)!)^{1/2} \\ &(y^\dagger)^k|n\rangle = |n+k\rangle((n+k)!/n!)^{1/2} \end{aligned} \tag{21}$$

are eigenstates of the seed. In particular,

$$\begin{aligned} 2\exp(2\mathrm{i}p; q)|n\rangle &= 2\exp(-2y^\dagger; y)|n\rangle \\ &= 2\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}(y^\dagger)^k y^k|n\rangle \\ &= |n\rangle 2\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-2)^k \\ &= |n\rangle 2(-1)^n \end{aligned} \tag{22}$$

which identifies the eigenvalues $2(-1)^n$ and tells us that

$$2\exp(2\mathrm{i}p; q) = \sum_{n=0}^{\infty}|n\rangle 2(-1)^n\langle n| = 2(-1)^{y^\dagger y}. \tag{23}$$

Incidentally, we observe that the analogous evaluation of

$$2\exp(2y; y^\dagger)|n\rangle = |n\rangle\sum_{k=0}^{\infty}\frac{(n+k)!}{k!\,n!}2^k \tag{24}$$

results in a divergent series, which justifies the assertion after (15).

The spectrum of the seed is highly degenerate; for even values of n the eigenvalue is 2 and for odd values it is -2. Therefore, any state with a definite parity under the reflection

$$q \to -q \qquad p \to -p \tag{25}$$

is an eigenstate of the seed, twice the parity being the eigenvalue. In other words, the similarity transformation associated with the seed,

$$\begin{Bmatrix} q \\ p \end{Bmatrix} 2\exp(2\mathrm{i}p; q) = 2\exp(2\mathrm{i}p; q)\begin{Bmatrix} -q \\ -p \end{Bmatrix} \tag{26}$$

is the space reflection (25). This fact seems not to be on record. Note that the seed is not unitary; its square is 4, as follows immediately from (23).

The overall factor of 2, which prevents the seed from being unitary, is needed for the normalisation to unit trace:

$$\begin{aligned} \mathrm{Tr}\{2\exp[2\mathrm{i}(p-p'); (q-q')]\} &= \mathrm{Tr}[2\exp(2\mathrm{i}p; q)] \\ &= \int\frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi}2\exp(2\mathrm{i}p''q'') = 1. \end{aligned} \tag{27}$$

When applied to (6), this has the consequence

$$\mathrm{Tr}\,F(q,p) = \int\frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi}F_W(q',p') \tag{28}$$

which is a basic property of the Wigner function. A related statement is

$$\mathrm{Tr}\{2\exp[2\mathrm{i}(p-p');(q-q')]2\exp[2\mathrm{i}(p-p'');(q-q'')]\}=2\pi\delta(q'-q'')\delta(p'-p'') \quad (29)$$

which expresses both the orthogonality and the completeness of the basis. Its fundamental implication

$$\mathrm{Tr}[F(q,p)G(q,p)]=\int\frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi}F_{\mathrm{W}}(q',p')G_{\mathrm{W}}(q',p') \quad (30)$$

can be regarded as one of the defining properties of the Wigner function [6].

When integrated along a straight line in phase space, the Wigner basis yields the projection operator for the orthogonal direction:

$$\begin{aligned}\int \mathrm{d}s\, 2\exp[2\mathrm{i}(p-p'-s\cos\phi);(q-q'+s\sin\phi)]\\ =2\pi\delta[(q-q')\cos\phi+(p-p')\sin\phi]\end{aligned} \quad (31)$$

specific examples being

$$\begin{aligned}&\int \mathrm{d}p'\,2\exp[2\mathrm{i}(p-p');(q-q')]=2\pi\delta(q-q')\\ &\int \mathrm{d}q'\,2\exp[2\mathrm{i}(p-p');(q-q')]=2\pi\delta(p-p').\end{aligned} \quad (32)$$

Because of the translational and rotational invariance it suffices to prove the last statement, which is done by evaluating, for example, the $\langle p''|, |q''\rangle$ matrix element on both sides. Immediate consequences of (32) are

$$\begin{aligned}&\int\frac{\mathrm{d}q'}{2\pi}F_{\mathrm{W}}(q',p')=\langle p'|F(q,p)|p'\rangle\\ &\int\frac{\mathrm{d}p'}{2\pi}F_{\mathrm{W}}(q',p')=\langle q'|F(q,p)|q'\rangle\end{aligned} \quad (33)$$

and the corresponding statement to (31). Another implication is that, for any function f and arbitrary complex numbers α, β,

$$\text{if} \qquad F(q,p)=f(\alpha q+\beta p) \qquad \text{then} \qquad F_{\mathrm{W}}(q',p')=f(\alpha q'+\beta p') \quad (34)$$

which, in this generality, seems to be a new observation.

The list of properties of the Hermitian Wigner basis would be incomplete without mentioning its relation to Weyl's unitary basis [12, 13], which consists of the translators $\exp[\mathrm{i}(pq'-p'q)]$. Since

$$\mathrm{Tr}\{\exp[\mathrm{i}(pq'-p'q)]2\exp[2\mathrm{i}(p-p'');(q-q'')]\}=\exp[\mathrm{i}(p''q'-p'q'')] \quad (35)$$

we find according to (6) that

$$\exp[\mathrm{i}(pq'-p'q)]=\int\frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi}\exp[\mathrm{i}(p''q'-p'q'')]2\exp[2\mathrm{i}(p-p'');(q-q'')] \quad (36)$$

which is a two-dimensional Fourier transformation, with the inverse

$$2\exp[2\mathrm{i}(p-p');(q-q')]=\int\frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi}\exp[-\mathrm{i}(p'q''-p''q')]\exp[\mathrm{i}(pq''-p''q)]. \quad (37)$$

It is well known that the integral on the right-hand side constitutes the operator basis underlying Wigner's phase space function; the explicit expression on the left is new. In this integral form, the Wigner basis has been studied by a number of people (a recent review is [9]) who in presenting their results created the misleading notion that the Fourier integral (37) is the most useful (if not the only) way of writing the Wigner basis. I disagree with this view; the ordered exponential, paired with the invariance (10), enables one to simplify calculations enormously, as we shall illustrate with examples below, in addition to the derivation of (20).

There is one purpose for which the connection with Weyl's unitary basis is essential, namely the generalisation to discrete degrees of freedom (like spin, for instance). Then there are only a finite number of physically different states, and operators with properties analogous to q and p are not available. The unitary Weyl operators, in contrast, possess counterparts in the form of operators that permute the states cyclically [13]. We leave this aspect of the development to a separate publication [14].

The Wigner basis is clearly distinguished from all others by the invariance property (10). As a consequence, the resulting equations of motion for Wigner's phase space function are as close to classical dynamics as they can possibly be. We proceed by recalling that Heisenberg's equations of motion

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}-\frac{\partial}{\partial t}\right)F(q(t),p(t),t)=\frac{1}{\mathrm{i}}[F(q(t),p(t),t),H(q(t),p(t),t)] \tag{38}$$

are identical to the classical Hamilton equations, if the commutator $(1/\mathrm{i})[F, H]$ is equal to the (symmetrised) Poisson bracket. For an arbitrary operator F this requires that the (not necessarily Hermitian) Hamilton operator is bilinear in q and p, i.e. H is a sum of terms proportional to q, p, q^2, p^2, qp, and pq, with arbitrary numerical coefficient, but higher-order terms like q^3 or qp^2 are not present in H. Such a Hamiltonian operator implies linear equations of motion for $q(t)$ and $p(t)$, so that they are linearly related to their values at an earlier time. Thus,

$$\begin{aligned} q(t)&=q'(t)+\alpha(t)q(0)+\beta(t)q(0)\\ p(t)&=p'(t)+\sigma(t)q(0)+\tau(t)p(0)\end{aligned} \tag{39}$$

which, except for the innocuous translations by the numerical amounts $q'(t)$ and $p'(t)$, are of the form (9). Because of (10), the basis operators are simply translated as time goes by

$$2\exp[2\mathrm{i}(p(t)-p'');(q(t)-q'')]=2\exp[2\mathrm{i}(p(0)-p''_0(t));(q(0)-q''_0(t))] \tag{40}$$

where

$$\begin{aligned} q''_0(t)&=\tau(t)(q''-q'(t))-\beta(t)(p''-p'(t))\\ p''_0(t)&=-\sigma(t)(q''-q'(t))+\alpha(t)(p''-p'(t)).\end{aligned} \tag{41}$$

The basis as a whole remains unchanged.

The time-dependent Wigner function

$$\begin{aligned} F_{\mathrm{W}}(q',p',t)&=\mathrm{Tr}\{F(q(t),p(t),t)2\exp[2\mathrm{i}(p(t)-p');(q(t)-q')]\}\\ &=\mathrm{Tr}\{F(q(0),p(0),t)2\exp[2\mathrm{i}(p(0)-p');(q(0)-q')]\}\end{aligned} \tag{42}$$

is actually constant in time, unless the operator has an explicit time dependence. To make contact with classical phase space concepts, let us consider the density operator

$\rho(q(t), p(t), t)$ which specifies the system and is defined by the initial conditions. The fundamental properties of ρ are that it is Hermitian, its trace is unity and all expectation values (and therefore all eigenvalues) are non-negative. If the system is in a pure state then ρ is the corresponding projection operator. Since ρ is intrinsically time independent, $\mathrm{d}\rho/\mathrm{d}t = 0$, we have

$$\frac{\partial}{\partial t}\rho_{\mathrm{W}}(q', p', t) = -\frac{1}{\mathrm{i}}[\rho, H]_{\mathrm{W}}(q', p', t) \tag{43}$$

with obvious notation. Thus we are faced with finding the Wigner function of a commutator in terms of the individual Wigner functions. Of course, the solution of this problem is well known; a typical standard derivation that relies heavily upon the Fourier integral of (37) can be found in [9]; another one, based entirely on (1) is reproduced in [4]. Here we supply an argument which uses the ordered exponentials (7) directly.

The essential information necessary for expressing the Wigner function of a product of two operators, say $F(q, p)G(q, p)$, in terms of $F_{\mathrm{W}}(q', p')$ and $G_{\mathrm{W}}(q', p')$ is the Wigner function of the product of two basis operators. Thus, our attention is drawn to

$$\begin{aligned}
&\mathrm{Tr}\{2\exp[2\mathrm{i}(p-p'); (q-q')]2\exp[2\mathrm{i}(p-p''); (q-q'')]2\exp[2\mathrm{i}(p-p'''); (q-q''')]\} \\
&\quad = \int \mathrm{d}\tilde{q}'\,\mathrm{d}\tilde{p}'\,\mathrm{d}\tilde{q}''\,\mathrm{d}\tilde{p}''\langle\tilde{p}'|2\exp[2\mathrm{i}(p-p'); (q-q')]|\tilde{q}'\rangle \\
&\qquad \times\langle\tilde{q}'|2\exp[-2\mathrm{i}(q-q''); (p-p'')]|\tilde{p}''\rangle \\
&\qquad \times\langle\tilde{p}''|2\exp[2\mathrm{i}(p-p'''); (q-q''')]|\tilde{q}''\rangle\langle\tilde{q}''|\tilde{p}'\rangle
\end{aligned} \tag{44}$$

which, in view of the basic property (3) of the ordered exponentials, is equal to

$$\begin{aligned}
&\int \frac{\mathrm{d}\tilde{q}'\,\mathrm{d}\tilde{p}'}{2\pi}\frac{\mathrm{d}\tilde{q}''\,\mathrm{d}\tilde{p}''}{2\pi}2^3\exp[2\mathrm{i}(\tilde{p}'-p')(\tilde{q}'-q') - \mathrm{i}\tilde{p}'\tilde{q}' - 2\mathrm{i}(\tilde{q}'-q'')(\tilde{p}''-p'') + \mathrm{i}\tilde{q}'\tilde{p}'' \\
&\qquad + 2\mathrm{i}(\tilde{p}''-p''')(\tilde{q}''-q''') - \mathrm{i}\tilde{p}''\tilde{q}'' + \mathrm{i}\tilde{q}''\tilde{p}'] \\
&\quad = 4\exp[-2\mathrm{i}p'(q''-q''') - 2\mathrm{i}p''(q'''-q') - 2\mathrm{i}p'''(q'-q'')].
\end{aligned} \tag{45}$$

With this, the known results

$$\begin{aligned}
(FG)_{\mathrm{W}}(q', p') &= \int \frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi}\frac{\mathrm{d}q'''\,\mathrm{d}p'''}{2\pi}4\exp[-2\mathrm{i}p'(q''-q''') - 2\mathrm{i}p''(q'''-q') - 2\mathrm{i}p'''(q'-q'')] \\
&\quad \times F_{\mathrm{W}}(q'', p'')G_{\mathrm{W}}(q''', p''') \\
&= \exp\left(\frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial q''}\frac{\partial}{\partial p'''} - \frac{\partial}{\partial q'''}\frac{\partial}{\partial p''}\right)\right)F_{\mathrm{W}}(q'', p'')G_{\mathrm{W}}(q''', p''')\Big|_{\substack{q''=q'''=q' \\ p''=p'''=p'}} \\
&\equiv \exp\left(\frac{\mathrm{i}}{2}\left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial p'}\right\}\right)(F_{\mathrm{W}}G_{\mathrm{W}})(q', p')
\end{aligned} \tag{46}$$

are reproduced immediately. The last version of (46) introduces the differential operator of the classical Poisson bracket for suggestive notational simplification (a similar notation is used in [15]). As a consequence, the Wigner function of a commutator is

$$\frac{1}{\mathrm{i}}[F, G]_{\mathrm{W}}(q', p') = 2\sin\left(\frac{1}{2}\left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial p'}\right\}\right)(F_{\mathrm{W}}G_{\mathrm{W}})(q', p'). \tag{47}$$

Equation (43) then appears as

$$\frac{\partial}{\partial t}\rho_{\rm W}(q', p', t) = -2\sin\left(\frac{1}{2}\left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial p'}\right\}\right)(\rho_{\rm W}H_{\rm W})(q', p', t), \tag{48}$$

where only odd powers of derivatives (of $\rho_{\rm W}$ or $H_{\rm W}$) are present. This is formally identical to Liouville's theorem if the sine function can be equivalently replaced by its argument. Such is the situation, for arbitrary ρ, if $H_{\rm W}(q', p')$ is bilinear in q' and p', which in turn implies that $H(q, p)$ is bilinear. This closes the argument: whenever Heisenberg's equations of motion are (formally) identical to the classical Hamilton equations, the Wigner function of the density operator obeys the classical Liouville theorem. Indeed, Wigner's phase space description is as close to classical physics as it possibly can be. This property is not shared by other phase space functions referring to different bases. Examples are given in the subsequent sections.

A first demonstration of the power of our new approach is the derivation of (20) given above. As a second we choose the time evolution operator for a particle under a constant force k,

$$\begin{aligned} F(q, p) &= \exp[-\mathrm{i}(\tfrac{1}{2}p^2 - kq)t] \\ &= \exp[\mathrm{i}kqt]\exp[-\tfrac{1}{2}\mathrm{i}(p + \tfrac{1}{2}kt)^2 t]\exp[-\mathrm{i}k^2t^3/24]. \end{aligned} \tag{49}$$

Its Wigner function,

$$\begin{aligned} F_{\rm W}(q', p') &= \mathrm{Tr}\{F(q, p)2\exp[2\mathrm{i}(p - p'); (q - q')]\} \\ &= \int \frac{\mathrm{d}q''\,\mathrm{d}p''}{2\pi}\exp(\mathrm{i}kq''t)\exp[-\tfrac{1}{2}\mathrm{i}(p'' + \tfrac{1}{2}kt)^2 t]\exp(-\mathrm{i}k^2t^3/24) \\ &\quad \times 2\exp[2\mathrm{i}(p'' - p')(q'' - q')] \end{aligned} \tag{50}$$

is obtained immediately whereby, once again, the ordered forms of the operators provide an enormous simplification. Upon evaluating this elementary integral we find

$$F_{\rm W}(q', p') = \exp[-\mathrm{i}(\tfrac{1}{2}p'^2 - kq')t]\exp(-\mathrm{i}k^2t^3/24). \tag{51}$$

Naturally, the second exponential factor constitutes the quantum corrections to the first, which is of classical appearance. It is worth mentioning that the Airy function $\mathrm{Ai}(x)$, with the defining property

$$\exp(-\mathrm{i}z^3/3) = \int \mathrm{d}x\,\mathrm{Ai}(x)\exp(\mathrm{i}xz) \tag{52}$$

can be used to rewrite (51):

$$F_{\rm W}(q', p') = \int \mathrm{d}x\,\mathrm{Ai}(x)\exp[-\mathrm{i}(\tfrac{1}{2}p'^2 - kq' - \tfrac{1}{2}x|k|^{2/3})t]. \tag{53}$$

The exponent is now linear in t, as is the original one in (49). Consequently, Fourier transformation in t implies that the Wigner function of any operator of the form $F(q, p) = f(\frac{1}{2}p^2 - kq)$ is given by

$$\int \mathrm{d}x\,\mathrm{Ai}(x)f(\tfrac{1}{2}p'^2 - kq' - \tfrac{1}{2}x|k|^{2/3}). \tag{54}$$

Such Airy averages have been employed successfully in studying quantum corrections to semiclassical models in atomic physics ([16] or, for more details, [17]).

The third illustrative example is the density operator of a harmonic oscillator at thermal equilibrium,

$$\rho(q, p) = [1 - \exp(-\beta)] \exp(-\beta y^\dagger y)$$
$$= \gamma \exp(-\gamma y^\dagger; y) \tag{55}$$

where $\beta > 0$, $\gamma = 1 - \mathrm{e}^{-\beta}$, and $y, y^\dagger$ are the non-Hermitian dynamical variables of (14). Of course, we utilise the invariance property (10) and follow the strategy that led to (20), i.e. first we evaluate the Wigner function of $\gamma \exp(\mathrm{i}\gamma p; q)$,

$$\mathrm{Tr}\{\gamma \exp(\mathrm{i}\gamma p; q) 2 \exp[-2\mathrm{i}(q-q'); (p-p')]\} = \frac{2\gamma}{2-\gamma} \exp\left(\mathrm{i}\frac{2\gamma}{2-\gamma} p'q'\right) \tag{56}$$

and then perform the replacements $q' \to y' = (q' + \mathrm{i}p')/\sqrt{2}$, $p' \to \mathrm{i}y^{\dagger\prime} = (p' + \mathrm{i}q')/\sqrt{2}$. The outcome is

$$\rho_{\mathrm{W}}(q', p') = \frac{2\gamma}{2-\gamma} \exp\left(-\frac{\gamma}{2-\gamma}(q'^2 + p'^2)\right)$$
$$= 2 \tanh(\beta/2) \exp[-(q'^2 + p'^2) \tanh(\beta/2)]. \tag{57}$$

The Wigner functions of the projection operators to the stationary uncertainty states $|n\rangle$

$$\rho^{(n)}(q, p) = |n\rangle\langle n| \tag{58}$$

are the last example. They can easily be extracted from (57). Since

$$\rho(q, p) = [1 - \exp(-\beta)] \sum_{n=0}^{\infty} \exp(-n\beta) \rho^{(n)}(q, p) \tag{59}$$

(57) implies the relation

$$\sum_{n=0}^{\infty} z^n \rho_{\mathrm{W}}^{(n)}(q', p') = 2 \exp[-(q'^2 + p'^2)] \frac{1}{1+z} \exp\left(\frac{2z}{1+z}(q'^2 + p'^2)\right)$$
$$= 2 \exp[-(q'^2 + p'^2)] \sum_{n=0}^{\infty} (-z)^n L_n(2(q'^2 + p'^2)) \tag{60}$$

where $z = \mathrm{e}^{-\beta}$ and we have recognised one of the generating functions of the Laguerre polynomials. Thus we find

$$\rho_{\mathrm{W}}^{(n)}(q', p') = 2(-1)^n \exp[-(q'^2 + p'^2)] L_n(2(q'^2 + p'^2)). \tag{61}$$

The power of the new formalism is strikingly revealed as soon as one compares this very brief derivation of (57) and (61) with the corresponding equations (2.90)-(2.115) of [4].

3. Kirkwood-type scale-invariant bases

On either side of (15) we can replace the factors of 2 by another real number, λ. In this section we shall deal with the scale-invariant bases characterised by the seeds that correspond to the left-hand side of (15). Equivalent expressions are

$$\lambda \exp(\mathrm{i}\lambda p; q) = \begin{cases} \bar{\lambda} \exp(-\mathrm{i}\bar{\lambda} q; p) & \text{for } \lambda > 1 \\ 2\pi\delta(q)\delta(p) & \text{for } \lambda = 1, \bar{\lambda} = \infty \end{cases} \tag{62}$$

where $\bar{\lambda}=\lambda/(\lambda-1)$. For notational simplicity, we do not consider $\lambda<1$, to which values the equations can be extended without particular complications. In the limiting situation $\bar{\lambda}=1$, $\lambda=\infty$, we have

$$2\pi\delta(p)\delta(q)=\exp(-iq;p). \tag{63}$$

Incidentally, we remark that the $\lambda=1$ seed has already occurred in (18). It is the seed of the basis underlying Kirkwood's phase space function [7]. The one with $\lambda=2$ is the Wigner seed, of course.

Whereas the seeds (62) are evidently scale invariant and of unit trace, they are, for $\lambda\neq 2$, neither Hermitian nor rotationally invariant. The former implies that Hermitian operators are not represented by real functions, as they are in Wigner's description; the latter means that different directions in phase space are not equivalent.

As (62) and (63) show, taking the adjoint is tantamount to replacing λ by $\bar{\lambda}$, and vice versa. Indeed, their relation can be written in the symmetrical forms

$$(\lambda-1)(\bar{\lambda}-1)=1 \qquad \lambda\bar{\lambda}=\lambda+\bar{\lambda} \qquad (1/\lambda)+(1/\bar{\lambda})=1. \tag{64}$$

The λ and the $\bar{\lambda}$ bases are further connected by the orthogonality-completeness relation

$$\begin{aligned}\mathrm{Tr}\{\lambda\ &\exp[\mathrm{i}\lambda(p-p');(q-q')]\bar{\lambda}\ \exp[\mathrm{i}\bar{\lambda}(p-p'');(q-q'')]\}\\ &=\mathrm{Tr}\{\lambda\ \exp[\mathrm{i}\lambda(p-p');(q-q')]\lambda\ \exp[-\mathrm{i}\lambda(q-q'');(p-p'')]\}\\ &=2\pi\delta(q'-q'')\delta(p'-p'')\end{aligned} \tag{65}$$

which is the analogue of (29) (and identical to that equation if $\lambda=\bar{\lambda}=2$). The expansion of an arbitrary operator is, therefore,

$$F(q,p)=\int\frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi}F_{\mathrm{K},\lambda}(q',p')\lambda\ \exp[\mathrm{i}\lambda(p-p');(q-q')] \tag{66}$$

with the coefficient function

$$F_{\mathrm{K},\lambda}(q',p')=\mathrm{Tr}\{F(q,p)\lambda\ \exp[-\mathrm{i}\lambda(q-q');(p-p')]\}. \tag{67}$$

Since (65) pairs one basis with its adjoint, the statement corresponding to (30) is

$$\mathrm{Tr}[F^{\dagger}(q,p)G(q,p)]=\int\frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi}F^{*}_{\mathrm{K},\lambda}(q',p')G_{\mathrm{K},\lambda}(q',p') \tag{68}$$

for all λ.

After presenting (67) in the form analogous to (1)

$$F_{\mathrm{K},\lambda}(q',p')=\int\mathrm{d}s\langle q'-s/\bar{\lambda}|F(q,p)|q'+s/\lambda\rangle\exp(\mathrm{i}p's) \tag{69}$$

we observe that p' is still the momentum associated with the separation $s=q'''-q''$ of the two q values to which the matrix element $\langle q''|F(q,p)|q'''\rangle$ refers, whereas $q'=q''/\lambda+q'''/\bar{\lambda}$ is now a weighted average. In the Wigner description, $\lambda=\bar{\lambda}=2$, it is the point halfway between q'' and q'''. In particular, for $\lambda=1$, $\bar{\lambda}=\infty$ we have

$$F_{\mathrm{K},\lambda=1}(q',p')=\langle q'|F(q,p)|p'\rangle/\langle q'|p'\rangle. \tag{70}$$

and for $\lambda=\infty$, $\bar{\lambda}=1$ it is

$$F_{\mathrm{K},\lambda=\infty}(q',p')=\langle p'|F(q,p)|q'\rangle/\langle p'|q'\rangle. \tag{71}$$

These can be cast in the following way: put $F(q,p)$ in the $(q;p)$-ordered form, then replace q by q' and p by p'; this produces $F_{\mathrm{K},\lambda=1}(q',p')$; likewise $F_{\mathrm{K},\lambda=\infty}(q',p')$ is obtained by $(p;q)$ ordering of $F(q,p)$. Of course, inserting the delta functions of (62) and (63) into (67) establishes these procedures as well.

For (relatively) simple Hamilton operators, the similarity transformation characterising the λ basis, i.e.

$$\begin{aligned} q\lambda\,\exp(\mathrm{i}\lambda p;q) &= \lambda\,\exp(\mathrm{i}\lambda p;q)(1-\lambda)q \\ \lambda\,\exp(\mathrm{i}\lambda p;q)p &= (1-\lambda)p\lambda\,\exp(\mathrm{i}\lambda p;q) \end{aligned} \tag{72}$$

(see (26)) can be used, as a very efficient tool, for translating the operator equation of motion (38) into its numerical analogue. For illustration consider a free particle with $H=\frac{1}{2}p^2$. Combined with

$$p\lambda\,\exp[\mathrm{i}\lambda(p-p');(q-q')] = \left(p'+\frac{\mathrm{i}}{\lambda}\frac{\partial}{\partial q'}\right)\lambda\,\exp[\mathrm{i}\lambda(p-p');(q-q')] \tag{73}$$

(72) implies

$$\begin{aligned} &[\lambda\,\exp[\mathrm{i}\lambda(p-p');(q-q')],\tfrac{1}{2}p^2] \\ &\qquad=\frac{1}{2}\left[\left(p'-\frac{\mathrm{i}}{\bar\lambda}\frac{\partial}{\partial q'}\right)^2-\left(p'+\frac{\mathrm{i}}{\lambda}\frac{\partial}{\partial q'}\right)^2\right]\lambda\,\exp[\mathrm{i}\lambda(p-p');(q-q')] \\ &\qquad=\mathrm{i}\left(-p'+\frac{\mathrm{i}}{2}\frac{\lambda-\bar\lambda}{\lambda+\bar\lambda}\frac{\partial}{\partial q'}\right)\frac{\partial}{\partial q'}\lambda\,\exp[\mathrm{i}\lambda(p-p');(q-q')] \end{aligned} \tag{74}$$

which is a particular realisation of a general statement. We now insert $\rho(q,p)$ in the form (66) into (38), recalling that $\mathrm{d}\rho/\mathrm{d}t=0$, employ (74), and perform two partial integrations over q', to arrive at

$$\frac{\partial}{\partial t}\rho_{\mathrm{K},\lambda}(q',p',t) = -\left(p'+\frac{\mathrm{i}}{2}\frac{\lambda-\bar\lambda}{\lambda+\bar\lambda}\frac{\partial}{\partial q'}\right)\frac{\partial}{\partial q'}\rho_{\mathrm{K},\lambda}(q',p',t). \tag{75}$$

This differs from the corresponding classical Liouville equation, unless $\lambda=\bar\lambda=2$. In particular, for $\lambda=1$, $\bar\lambda=\infty$ we have $(\lambda-\bar\lambda)/(\lambda+\bar\lambda)=-1$ and for $\lambda=\infty$, $\bar\lambda=1$ we have $(\lambda-\bar\lambda)/(\lambda+\bar\lambda)=+1$. Even for a free particle, the dynamics of Kirkwood-type phase space functions differs significantly from classical dynamics.

Let us close the discussion of the Kirkwood-type bases by pointing out that equations (33) stay valid when $F_{\mathrm{W}}(q',p')$ is replaced by $F_{\mathrm{K},\lambda}(q',p')$. Thus, as conjectured by Kirkwood, one can indeed 'construct a number of different functions of p' and q', all of which ... give the correct results for the momentum and configuration probabilities after integration over p' or q'' [7].

4. Glauber-type rotationally invariant bases

Upon replacing the factors of 2 by λ on the right-hand side of (15), we get the Hermitian rotationally invariant seeds

$$\lambda\,\exp(-\lambda y^+;y) = \sum_{n=0}^{\infty}|n\rangle\lambda(1-\lambda)^n\langle n| = \lambda(1-\lambda)^{y^\dagger y} \tag{76}$$

where the alternative forms are immediate generalisations of (23). Equivalent $(q; p)$- and $(p; q)$-ordered expressions are (with λ and $\bar{\lambda}$ as in (64))

$$\lambda \exp(-\lambda y^+; y)$$
$$= \left(2\frac{\lambda^2\bar{\lambda}^2}{\lambda^2+\bar{\lambda}^2}\right)^{1/2} \exp\left(-\frac{1}{2}\frac{\bar{\lambda}^2-\lambda^2}{\bar{\lambda}^2+\lambda^2}q^2\right)$$
$$\times \exp\left(-\mathrm{i}\frac{\lambda^2\bar{\lambda}^2}{\lambda^2+\bar{\lambda}^2}q; p\right) \exp\left(-\frac{1}{2}\frac{\bar{\lambda}^2-\lambda^2}{\bar{\lambda}^2+\lambda^2}p^2\right)$$
$$= \text{the same expression with } q \to p, p \to -q. \tag{77}$$

As a matter of fact, these versions are only formally equivalent. It is true that they all give the same value to the $\langle y^{+\prime}|, |y''\rangle$ matrix element, but in other respects they are not equally good. Consider, for example, the trace of the seed. If it is evaluated in terms of the eigenstates of y and $y^\dagger$ (their basic properties being

$$y|y'\rangle = |y'\rangle y' \qquad \langle y^{\dagger\prime}|y^\dagger = y^{\dagger\prime}\langle y^{\dagger\prime}|$$
$$(|y'\rangle)^\dagger = \langle y^{\dagger\prime} = (y')^*| \qquad \langle y^{\dagger\prime}|y''\rangle = \exp(y^{\dagger\prime}y'') \tag{78}$$
$$\int \frac{(\mathrm{d}y')}{2\pi}|y'\rangle \exp(-y^{\dagger\prime}y')\langle y^{\dagger\prime}| = 1$$

where $\mathrm{d}y' = \mathrm{d}p'\,\mathrm{d}q'$ if $y' = (q'+\mathrm{i}p')/\sqrt{2} = (y^{\dagger\prime})^*$ in the integrand†; this reflects $\alpha = \tau = 1/\sqrt{2}$, $\beta = \sigma = \mathrm{i}/\sqrt{2}$, $C_{qQ} = C_{P_p} = \pi^{-1/4}$ in (9), (11) and (12)) then one obtains

$$\int \frac{(\mathrm{d}y')}{2\pi}\lambda \exp[-\lambda y^{\dagger\prime}y'] = \int \frac{\mathrm{d}q'\,\mathrm{d}p'}{2\pi}\lambda \exp[-\tfrac{1}{2}\lambda(q'^2+p'^2)] = 1 \tag{79}$$

as expected. Whereas this integral converges for all $\lambda > 0$, the series that sums the eigenvalues,

$$\sum_{n=0}^{\infty} \lambda(1-\lambda)^n = \frac{\lambda}{1-(1-\lambda)} = 1 \tag{80}$$

diverges unless $|\lambda - 1| < 1$, i.e. $0 < \lambda < 2$. Likewise the phase space integral obtained directly from the $(q; p)$-ordered form (77) requires $\bar{\lambda}^2 \geq \lambda^2$ for convergence; thus, here we need $0 < \lambda \leq 2$.

Difficulties of this kind occur only for the Glauber-type seeds (76) but not for those of the Kirkwood type (62). The obvious reason is that the non-Hermitian operators y and $y^\dagger$ are not quite on an equal footing with the Hermitian q and p, because y has only right eigenstates and $y^\dagger$ only left ones. As a consequence, it is always possible to order an operator $(y^\dagger; y)$-wise whereas $(y; y^\dagger)$ orderings are available for a restricted class of operators only‡.

Concerning the seeds (76) the significance of this remark is that only the seeds with $\lambda \geq 2$ generate a basis, whereas the set of operators

$$\exp[-\mathrm{i}(pq'-p'q)]\lambda \exp(-\lambda y^\dagger; y) \exp[\mathrm{i}(pq'-p'q)]$$
$$= \exp(y^\dagger y' - y^{\dagger\prime}y)\lambda \exp(-\lambda y^\dagger; y) \exp(-y^\dagger y' + y^{\dagger\prime}y)$$
$$= \lambda \exp[-\lambda(y^\dagger - y^{\dagger\prime}); (y-y')] \tag{81}$$

† It should be noted that this is only one way, the standard one, of parametrising the $(\mathrm{d}y')$ integral. More generally the independent complex variables y' and $\mathrm{i}y^{\dagger\prime}$ are to be integrated along orthogonal contours. For details see [18].

‡ Conditions under which operators can be expanded into $y^j\,(y^\dagger)^k$ power series are formulated in [19].

is incomplete for $\lambda<2$. To make this point, we proceed from the analogue of (66) and (67),

$$F(q,p)=\int\frac{(\mathrm{d}y')}{2\pi}\,F_{G,\lambda}(y',y^{\dagger\prime})\lambda\,\exp[-\lambda(y^\dagger-y^{\dagger\prime});(y-y')] \tag{82}$$

where the coefficient function is the Glauber-type phase space function

$$F_{G,\lambda}(y',y^{\dagger\prime})=\mathrm{Tr}\{F(q,p)\bar\lambda\,\exp[-\bar\lambda(y^\dagger-y^{\dagger\prime});(y-y')]\}. \tag{83}$$

Again the λ basis is paired with the $\bar\lambda$ basis, the relations (64) still holding. Thus, an operator can be expanded in the λ basis if its trace with the elements of the $\bar\lambda$ basis exists. Now consider the projection operator (16) which in $(y^\dagger;y)$-ordered form appears as

$$\rho^{(\mu)}(y,y^\dagger)=\sqrt{1-\mu\mu^*}\,\exp(-\tfrac{1}{2}\mu y^{\dagger 2})\exp(-y^\dagger;y)\exp(-\tfrac{1}{2}\mu^* y^2) \tag{84}$$

where the parameter μ is related to ϕ and z of (17) by $\mu=\exp(2\mathrm{i}\phi)\tanh z$, so that $|\mu|<1$. We insert this $\rho^{(\mu)}$ into the trace of (83) to find $\rho^{(\mu)}_{G,\lambda}$, and observe that the resulting integrals converge only if

$$|\mu|\leqslant(\lambda/\bar\lambda)^2=(\lambda-1)^2. \tag{85}$$

For $\lambda<2$, the right-hand side is less than one, in which situation it is possible to pick $|\mu|$ so large that the inequality (85) is not obeyed. The corresponding $\rho^{(\mu)}$, therefore, cannot be expanded in the λ basis of Glauber type. In particular, for $\lambda=1$, when the phase space function is the so-called Glauber P function, (85) requires $\mu=0$. In other words, there is no P function for the projectors to 'squeezed states' ($\mu\neq 0$).

Quite analogous to the prescriptions given after (71), we have now: to obtain $F_{G,\lambda=1}(y',y^{\dagger\prime})$ put $F(q,p)$ in $(y;y^\dagger)$-ordered from, then replace y by y' and $y^\dagger$ by $y^{\dagger\prime}$; likewise $F_{G,\lambda=\infty}(y',y^{\dagger\prime})$ is obtained by $y^\dagger$; y ordering. Not surprisingly, it is easy to find operators for which there is no $\lambda=1$ phase space function, because $(y;y^\dagger)$ ordering is not generally possible [19].

An example for an operator, which can be expanded in any Glauber-type basis, is the density operator (55). We find

$$\begin{aligned}\rho_{G,\lambda}(y',y^{\dagger\prime})&=\mathrm{Tr}\{\gamma\exp(-\gamma y^\dagger;y)\bar\lambda\,\exp[-\bar\lambda(y^\dagger-y^{\dagger\prime});(y-y')]\}\\&=\frac{\lambda\gamma}{\lambda-\gamma}\exp\left(-\frac{\lambda\gamma}{\lambda-\gamma}y^{\dagger\prime}y'\right)\end{aligned} \tag{86}$$

where we recall that $\gamma=1-\mathrm{e}^{-\beta}$, $\beta>0$, thus $0<\gamma<1$, and remark that $y^{\dagger\prime}y'=\tfrac{1}{2}(q'^2+p'^2)$. The analogue of (60), here

$$\begin{aligned}\sum_{n=0}^{\infty}z^n\rho^{(n)}_{G,\lambda}(y',y^{\dagger\prime})&=\frac{\lambda}{\lambda-1+z}\exp\left(-\frac{\lambda(1-z)}{\lambda-1+z}y^{\dagger\prime}y'\right)\\&=\bar\lambda\,\exp(-\bar\lambda y^{\dagger\prime}y')\sum_{n=0}^{\infty}\left(-\frac{\bar\lambda}{\lambda}z\right)^n L_n(\lambda\bar\lambda y^{\dagger\prime}y')\end{aligned} \tag{87}$$

identifies the phase space functions of $\rho^{(n)}=|n\rangle\langle n|$ as

$$\rho^{(n)}_{G,\lambda}(y',y^{\dagger\prime})=\bar\lambda\left(-\frac{\bar\lambda}{\lambda}\right)^n\exp(-\bar\lambda y^{\dagger\prime}y')L_n(\lambda\bar\lambda y^{\dagger\prime}y'). \tag{88}$$

In the two extreme situations $\lambda \to 1$, $\bar{\lambda} \to \infty$ and $\lambda \to \infty$, $\bar{\lambda} \to 1$, the argument of the Laguerre polynomial becomes arbitrarily large (unless $y^{\dagger\prime} y' = 0$, i.e. unless $q' = 0$ and $p' = 0$), so that it is effectively equal to its leading power, $L_n(x \gg 1) \simeq (-x)^n/n!$, implying

$$\rho^{(n)}_{G,\lambda}(y', y^{\dagger\prime})|_{\lambda \to 1 \text{ or } \infty} = \frac{1}{n!} \bar{\lambda}^{2n+1} (y^{\dagger\prime} y')^n \exp(-\bar{\lambda} y^{\dagger\prime} y')|_{\bar{\lambda} \to \infty \text{ or } 1}. \tag{89}$$

For $\lambda \to \infty$, this gives the expected result

$$\begin{aligned} \rho^{(n)}_{G,\lambda=\infty}(y', y^{\dagger\prime}) &= \langle y^{\dagger\prime} | n \rangle \langle n | y' \rangle / \langle y^{\dagger\prime} | y' \rangle \\ &= \frac{1}{n!} (y^{\dagger\prime} y')^n \exp(-y^{\dagger\prime} y') \end{aligned} \tag{90}$$

whereas for $\lambda \to 1$, (89) shows that there is no $\rho^{(n)}_{G,\lambda=1}$, except for $n = 0$ when

$$\rho^{(n=0)}_{G,\lambda=1}(y', y^{\dagger\prime}) = \bar{\lambda} \exp(-\bar{\lambda} y^{\dagger\prime} y')|_{\bar{\lambda} \to \infty} = 2\pi \delta(q') \delta(p'). \tag{91}$$

This typical breakdown of the Glauber P representation (see also [20]) is more immediately recognised upon setting $\lambda = 1$ in (87), which produces

$$\begin{aligned} \sum_{n=0}^{\infty} z^n \rho^{(n)}_{G,\lambda=1}(y', y^{\dagger\prime}) &= \frac{1}{z} \exp\left[-\left(\frac{1}{z} - 1\right) y^{\dagger\prime} y' \right] \\ &= \exp(y^{\dagger\prime} y') \sum_{m=-\infty}^{-1} \frac{(-y^{\dagger\prime} y')^{-m-1}}{(-m-1)!} z^m \end{aligned} \tag{92}$$

in which positive powers of z on one side of the equation simply do not go together with negative powers on the other side.

Our final remark about the Glauber-type phase space functions concerns the differential equation obeyed by the density operator of a free particle. Proceeding from the analogue of (72)

$$\begin{aligned} y \lambda \exp(-\lambda y^{\dagger}; y) &= \lambda \exp(-\lambda y^{\dagger}; y)(1 - \lambda) y \\ \lambda \exp(-\lambda y^{\dagger}; y) y^{\dagger} &= (1 - \lambda) y^{\dagger} \lambda \exp(-\lambda y^{\dagger}; y) \end{aligned} \tag{93}$$

we find, by the same reasoning that resulted in (75),

$$\begin{aligned} \frac{\partial}{\partial t} \rho_{G,\lambda}(y', y^{\dagger\prime}, t) &= -\left\{ \frac{i}{2} (y^{\dagger\prime} - y') \left(\frac{\partial}{\partial y^{\dagger\prime}} + \frac{\partial}{\partial y'} \right) \right. \\ &\quad \left. - \frac{i}{4} \frac{\lambda - \bar{\lambda}}{\lambda + \bar{\lambda}} \left[\left(\frac{\partial}{\partial y^{\dagger\prime}} \right)^2 - \left(\frac{\partial}{\partial y'} \right)^2 \right] \right\} \rho_{G,\lambda}(y', y^{\dagger\prime}, t) \end{aligned} \tag{94}$$

or, expressed in terms of $q' = (y^{\dagger\prime} + y')/\sqrt{2}$ and $p' = i(y^{\dagger\prime} - y')/\sqrt{2}$,

$$\frac{\partial}{\partial t} \rho_{G,\lambda}(q', p', t) = -\left(p' + \frac{1}{2} \frac{\lambda - \bar{\lambda}}{\lambda + \bar{\lambda}} \frac{\partial}{\partial p'} \right) \frac{\partial}{\partial q'} \rho_{G,\lambda}(q', p', t). \tag{95}$$

Like (75), this is the corresponding classical Liouville equation only if $\lambda = \bar{\lambda} = 2$, when we are dealing with the Wigner function. For $\lambda = \infty$, $\bar{\lambda} = 1$ this equation was also obtained by O'Connell and Wigner [21] in a related context and by a very different method.

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